9-6 Exponential Growth and Decay

Main Ideas

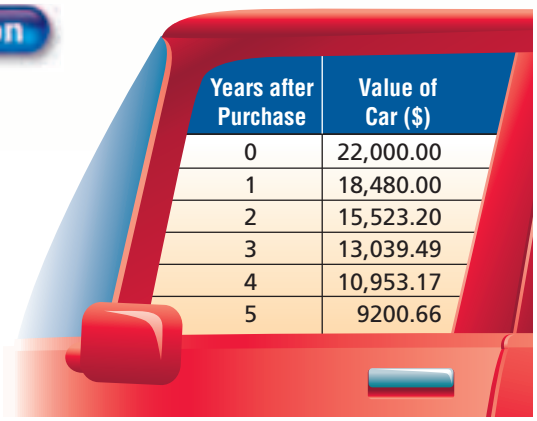
- Use logarithms to solve problems involving exponential decay.
- Use logarithms to solve problems involving exponential growth.

New Vocabulary

rate of decay
rate of growth

GET READY for the Lesson

Certain assets, like homes, can *appreciate* or increase in value over time. Others, like cars, *depreciate* or decrease in value with time. Suppose you buy a car for $22,000 and the value of the car decreases by 16% each year. The table shows the value of the car each year for up to 5 years after it was purchased.



Years after Purchase	Value of Car ($)
0	22,000.00
1	18,480.00
2	15,523.20
3	13,039.49
4	10,953.17
5	9200.66

Exponential Decay The depreciation of the value of a car is an example of exponential decay. When a quantity *decreases* by a fixed percent each year, or other period of time, the amount y of that quantity after t years is given by $y = a(1 - r)^t$, where a is the initial amount and r is the percent of decrease expressed as a decimal. The percent of decrease r is also referred to as the **rate of decay**.

EXAMPLE 1 Exponential Decay of the Form $y = a(1 - r)^t$

CAFFEINE **A cup of coffee contains 130 milligrams of caffeine. If caffeine is eliminated from the body at a rate of 11% per hour, how long will it take for half of this caffeine to be eliminated?**

Explore The problem gives the amount of caffeine consumed and the rate at which the caffeine is eliminated. It asks you to find the time it will take for half of the caffeine to be eliminated.

Plan Use the formula $y = a(1 - r)^t$. Let t be the number of hours since drinking the coffee. The amount remaining y is half of 130 or 65.

Solve

$y = a(1 - r)^t$ Exponential decay formula

$65 = 130(1 - 0.11)^t$ Replace y with 65, a with 130, and r with 11% or 0.11.

$0.5 = (0.89)^t$ Divide each side by 130.

$\log 0.5 = \log (0.89)^t$ Property of Equality for Logarithms

$\log 0.5 = t \log (0.89)$ Power Property for Logarithms

$\frac{\log 0.5}{\log 0.89} = t$ Divide each side by log 0.89.

$5.9480 \approx t$ Use a calculator.

It will take approximately 6 hours.

Study Tip

Rate of Change
Remember to rewrite the rate of change as a decimal before using it in the formula.

Check Use the formula to find how much of the original 130 milligrams of caffeine would remain after 6 hours.

$$y = a(1 - r)^t \qquad \text{Exponential decay formula}$$
$$= 130(1 - 0.11)^6 \qquad \text{Replace } a \text{ with 130, } r \text{ with 0.11, and } t \text{ with 6.}$$
$$\approx 64.6 \qquad \text{Use a calculator.}$$

Half of 130 is 65, so the answer seems reasonable. Half of the caffeine will be eliminated from the body in about 6 hours.

CHECK Your Progress

1. **SHOPPING** A store is offering a clearance sale on a certain type of digital camera. The original price for the camera was $198. The price decreases 10% each week until all of the cameras are sold. How many weeks will it take for the price of the cameras to drop below half of the original price?

Another model for exponential decay is given by $y = ae^{-kt}$, where k is a constant. This is the model preferred by scientists. Use this model to solve problems involving radioactive decay. Radioactive decay is the decrease in the intensity of a radioactive material over time. Being able to solve problems involving radioactive decay allows scientists to use carbon dating methods.

EXAMPLE 2 Exponential Decay of the Form $y = ae^{-kt}$

PALEONTOLOGY **The *half-life* of a radioactive substance is the time it takes for half of the atoms of the substance to disintegrate. All life on Earth contains Carbon-14, which decays continuously at a fixed rate. The half-life of Carbon-14 is 5760 years. That is, every 5760 years half of a mass of Carbon-14 decays away.**

a. What is the value of k and the equation of decay for Carbon-14?

Let a be the initial amount of the substance. The amount y that remains after 5760 years is then represented by $\frac{1}{2}a$ or $0.5a$.

$$y = ae^{-kt} \qquad \text{Exponential decay formula}$$
$$0.5a = ae^{-k(5760)} \qquad \text{Replace } y \text{ with } 0.5a \text{ and } t \text{ with 5760.}$$
$$0.5 = e^{-5760k} \qquad \text{Divide each side by } a.$$
$$\ln 0.5 = \ln e^{-5760k} \qquad \text{Property of Equality for Logarithmic Functions}$$
$$\ln 0.5 = -5760k \qquad \text{Inverse Property of Exponents and Logarithms}$$
$$\frac{\ln 0.5}{-5760} = k \qquad \text{Divide each side by } -5760.$$
$$\frac{-0.6931472}{-5760} \approx k \qquad \text{Use a calculator.}$$
$$0.00012 \approx k \qquad \text{Simplify.}$$

The value of k for Carbon-14 is 0.00012. Thus, the equation for the decay of Carbon-14 is $y = ae^{-0.00012t}$, where t is given in years.

(continued on the next page)

Richard T. Nowitz/Photo Researchers

Real-World Career

Paleontologist

Paleontologists study fossils found in geological formations. They use these fossils to trace the evolution of plant and animal life and the geologic history of Earth.

For more information, go to algebra2.com.

CHECK Use the formula to find the amount of a sample remaining after 5760 years. Use an original amount of 1.

$y = ae^{-0.00012t}$ Original equation

$= 1e^{-0.00012(5760)}$ $a = 1$ and $t = 5760$

≈ 0.501 Use a calculator.

About half of the amount remains. The answer checks.

b. A paleontologist examining the bones of a woolly mammoth estimates that they contain only 3% as much Carbon-14 as they would have contained when the animal was alive. How long ago did the mammoth die?

Let a be the initial amount of Carbon-14 in the animal's body. Then the amount y that remains after t years is 3% of a or $0.03a$.

$y = ae^{-0.00012t}$ Formula for the decay of Carbon-14

$0.03a = ae^{-0.00012t}$ Replace y with $0.03a$.

$0.03 = e^{-0.00012t}$ Divide each side by a.

$\ln 0.03 = \ln e^{-0.00012t}$ Property of Equality for Logarithms

$\ln 0.03 = -0.00012t$ Inverse Property of Exponents and Logarithms

$\frac{\ln 0.03}{-0.00012} = t$ Divide each side by -0.00012.

$29{,}221 \approx t$ Use a calculator.

The mammoth lived about 29,000 years ago.

CHECK Your Progress

2. A specimen that originally contained 150 milligrams of Carbon-14 now contains 130 milligrams. How old is the fossil?

Exponential Growth When a quantity *increases* by a fixed percent each time period, the amount y of that quantity after t time periods is given by $y = a(1 + r)^t$, where a is the initial amount and r is the percent of increase expressed as a decimal. The percent of increase r is also referred to as the **rate of growth**.

Test-Taking Tip

To change a percent to a decimal, drop the percent symbol and move the decimal point two places to the left.

1.5% = 0.015

STANDARDIZED TEST EXAMPLE

3 In 1910, the population of a city was 120,000. Since then, the population has increased by 1.5% per year. If the population continues to grow at this rate, what will the population be in 2010?

A 138,000 **B** 531,845 **C** 1,063,690 **D** 1.4×10^{11}

Read the Test Item

You need to find the population of the city 2010 − 1910, or 100, years later. Since the population is growing at a fixed percent each year, use the formula $y = a(1 + r)^t$.

Real-World Link

The Indian city of Varanasi is the world's oldest continuously inhabited city.

Source: tourismofindia.com

Solve the Test Item

$y = a(1 + r)^t$ Exponential growth formula

$= 120{,}000(1 + 0.015)^{100}$ Replace a with 120,000, r with 0.015, and t with 2010 − 1910, or 100.

$= 120{,}000(1.015)^{100}$ Simplify.

$\approx 531{,}845.48$ Use a calculator.

The answer is B.

CHECK Your Progress

3. Home values in Millersport increase about 4% per year. Mr. Thomas purchased his home eight years ago for $122,000. What is the value of his home now?

F $\$1.36 \times 10^5$ **G** $126,880 **H** $166,965 **J** $175,685

Online Personal Tutor at algebra2.com

Another model for exponential growth, preferred by scientists, is $y = ae^{kt}$, where k is a constant. Use this model to find the constant k.

EXAMPLE Exponential Growth of the Form $y = ae^{kt}$

4 POPULATION **As of 2005, China was the world's most populous country, with an estimated population of 1.31 billion people. The second most populous country was India, with 1.08 billion. The populations of India and China can be modeled by $I(t) = 1.08e^{0.0103t}$ and $C(t) = 1.31e^{0.0038t}$, respectively. According to these models, when will India's population be more than China's?**

You want to find t, the number of years, such that $I(t) > C(t)$.

$I(t) > C(t)$

$1.08e^{0.0103t} > 1.31e^{0.0038t}$ Replace $I(t)$ with $1.08e^{0.0103t}$ and $C(t)$ with $1.31e^{0.0038t}$.

$\ln 1.08e^{0.0103t} > \ln 1.31e^{0.0038t}$ Property of Inequality for Logarithms

$\ln 1.08 + \ln e^{0.0103t} > \ln 1.31 + \ln e^{0.0038t}$ Product Property of Logarithms

$\ln 1.08 + 0.0103t > \ln 1.31 + 0.0038t$ Inverse Property of Exponents and Logarithms

$0.0065t > \ln 1.31 - \ln 1.08$ Subtract $0.0038t$ from each side.

$t > \dfrac{\ln 1.31 - \ln 1.08}{0.0065}$ Divide each side by 0.006.

$t > 29.70$ Use a calculator.

After 30 years, or in 2035, India will be the most populous country.

Concepts in Motion

Interactive Lab
algebra2.com

CHECK Your Progress

4. BACTERIA Two different types of bacteria in two different cultures reproduce exponentially. The first type can be modeled by $B_1(t) = 1200\ e^{0.1532t}$, and the second can be modeled by $B_2(t) = 3000\ e^{0.0466t}$, where t is the number of hours. According to these models, how many hours will it take for the amount of B_1 to exceed the amount of B_2?

CHECK Your Understanding

Example 1 (pp. 544–545)

1. **POLICE** Police use blood alcohol content (BAC) to measure the percent concentration of alcohol in a person's bloodstream. In most states, a BAC of 0.08 percent means a person is not allowed to drive. Each hour after drinking, a person's BAC may decrease by 15%. If a person has a BAC of 0.18, how many hours will he need to wait until he can legally drive?

Example 2 (pp. 545–546)

SPACE **For Exercises 2–4, use the following information.**
A radioisotope is used as a power source for a satellite. The power output P (in watts) is given by $P = 50\,e^{-\frac{t}{250}}$, where t is the time in days.

2. Is the formula for power output an example of exponential growth or decay? Explain your reasoning.
3. Find the power available after 100 days.
4. Ten watts of power are required to operate the equipment in the satellite. How long can the satellite continue to operate?

Example 3 (pp. 546–547)

5. **STANDARDIZED TEST PRACTICE** The weight of a bar of soap decreases by 2.5% each time it is used. If the bar weighs 95 grams when it is new, what is its weight to the nearest gram after 15 uses?

A 57.5 g **B** 59.4 g **C** 65 g **D** 93 g

Example 4 (p. 547)

POPULATION GROWTH **For Exercises 6 and 7, use the following information.**
Fayette County, Kentucky, grew from a population of 260,512 in 2000 to a population of 268,080 in 2005.

6. Write an exponential growth equation of the form $y = ae^{kt}$ for Fayette County, where t is the number of years after 2000.
7. Use your equation to predict the population of Fayette County in 2015.

Exercises

HOMEWORK HELP	
For Exercises	See Examples
8	1
9–11	2
12–14	3
15, 16	4

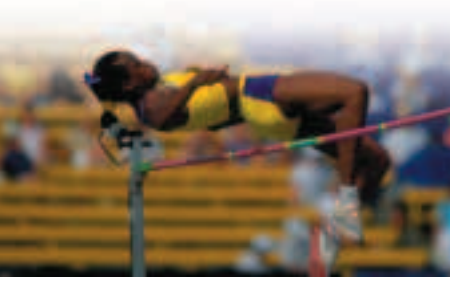
8. **COMPUTERS** Zeus Industries bought a computer for $2500. If it depreciates at a rate of 20% per year, what will be its value in 2 years?

9. **HEALTH** A certain medication is eliminated from the bloodstream at a steady rate. It decays according to the equation $y = ae^{-0.1625t}$, where t is in hours. Find the half-life of this substance.

10. **PALENTOLOGY** A paleontologist finds a bone of a human. In the laboratory, she finds that the Carbon-14 found in the bone is $\frac{2}{3}$ of that found in living bone tissue. How old is this bone?

11. **ANTHROPOLOGY** An anthropologist studying the bones of a prehistoric person finds there is so little remaining Carbon-14 in the bones that instruments cannot measure it. This means that there is less than 0.5% of the amount of Carbon-14 the bones would have contained when the person was alive. How long ago did the person die?

12. **REAL ESTATE** The Martins bought a condominium for $145,000. Assuming that the value of the condo will appreciate at most 5% a year, how much will the condo be worth in 5 years?



Karl Weatherly/CORBIS

Real-World Link

The women's high jump competition first took place in the USA in 1895, but it did not become an Olympic event until 1926.

Source: www.princeton.edu

ECONOMICS **For Exercises 13 and 14, use the following information.**
The annual Gross Domestic Product (GDP) of a country is the value of all of the goods and services produced in the country during a year. During the period 2001–2004, the Gross Domestic Product of the United States grew about 2.8% per year, measured in 2004 dollars. In 2001, the GDP was $9891 billion.

13. Assuming this rate of growth continues, what will the GDP of the United States be in the year 2015?

14. In what year will the GDP reach $20 trillion?

BIOLOGY **For Exercises 15 and 16, use the following information.**
Bacteria usually reproduce by a process known as *binary fission*. In this type of reproduction, one bacterium divides, forming two bacteria. Under ideal conditions, some bacteria reproduce every 20 minutes.

15. Find the constant k for this type of bacteria under ideal conditions.

16. Write the equation for modeling the exponential growth of this bacterium.

17. **OLYMPICS** In 1928, when the high jump was first introduced as a women's sport at the Olympic Games, the winning women's jump was 62.5 inches, while the winning men's jump was 76.5 inches. Since then, the winning jump for women has increased by about 0.38% per year, while the winning jump for men has increased at a slower rate, 0.3%. If these rates continue, when will the women's winning high jump be higher than the men's?

18. **HOME OWNERSHIP** The Mendes family bought a new house 10 years ago for $120,000. The house is now worth $191,000. Assuming a steady rate of growth, what was the yearly rate of appreciation?

FOOD **For Exercises 19 and 20, use the table of suggested times for cooking potatoes in a microwave oven. Assume that the number of minutes is a function of some power of the number of potatoes.**

Number of 8 oz. Potatoes	Cooking Time (min)
2	10
4	15

Source: wholehealthmd.com

EXTRA PRACTICE
See pages 911, 934.
MathOnline
Self-Check Quiz at algebra2.com

19. Write an equation in the form $t = an^b$, where t is the time in minutes, n is the number of potatoes, and a and b are constants. (*Hint:* Use a system of equations to find the constants.)

20. According to the formula, how long should you cook six 8-ounce potatoes in a microwave?

H.O.T. Problems

21. **REASONING** Explain how to solve $y = (1 + r)^t$ for t.

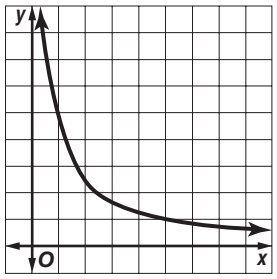
22. **OPEN ENDED** Give an example of a quantity that grows or decays at a fixed rate. Write a real-world problem involving the rate and solve by using logarithms.

23. **CHALLENGE** The half-life of radium is 1620 years. When will a 20-gram sample of radium be completely gone? Explain your reasoning.

24. ***Writing in Math*** Use the information about car values on page 544 to explain how you can use exponential decay to determine the current value of a car. Include a description of how to find the percent decrease in the value of the car each year and a description of how to find the value of a car for any given year when the rate of depreciation is known.

STANDARDIZED TEST PRACTICE

25. ACT/SAT The curve represents a portion of the graph of which function?



A $y = 50 - x$ **C** $y = e^{-x}$

B $y = \log x$ **D** $xy = 5$

26. REVIEW A radioactive element decays over time, according to the equation

$$y = x\left(\frac{1}{4}\right)^{\frac{t}{200}},$$

where x = the number of grams present initially and t = time in years. If 500 grams were present initially, how many grams will remain after 400 years?

F 12.5 grams **H** 62.5 grams

G 31.25 grams **J** 125 grams

Spiral Review

Write an equivalent exponential or logarithmic equation. (Lesson 9-5)

27. $e^3 = y$ **28.** $e^{4n-2} = 29$ **29.** $\ln 4 + 2 \ln x = 8$

Solve each equation or inequality. Round to four decimal places. (Lesson 9-4)

30. $16^x = 70$ **31.** $2^{3p} > 1000$ **32.** $\log_b 81 = 2$

BUSINESS **For Exercises 33–35, use the following information.**
A small corporation decides that 8% of its profits would be divided among its six managers. There are two sales managers and four nonsales managers. Fifty percent would be split equally among all six managers. The other 50% would be split among the four nonsales managers. Let p represent the profits. (Lesson 8-2)

33. Write an expression to represent the share of the profits each nonsales manager will receive.

34. Simplify this expression.

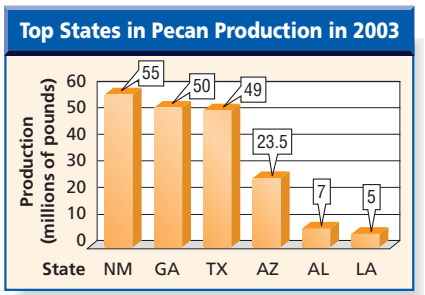
35. Write an expression in simplest form to represent the share of the profits each sales manager will receive.

AGRICULTURE **For Exercises 36–38, use the graph at the right.**
U.S. growers were forecasted to produce 264 million pounds of pecans in 2003. (Lesson 6-1)

36. Write the number of pounds of pecans forecasted by U.S. growers in 2003 in scientific notation.

37. Write the number of pounds of pecans produced by Georgia in 2003 in scientific notation.

38. What percent of the overall pecan production for 2003 can be attributed to Georgia?



Source: www.nass.usda.gov